13. BOLOTIN V.V., On the density of parametric resonances, PMM, Vol.44, No.6, 1980.

Translated by M.D.F.

PMM U.S.S.R.,Vol.48,No.5,pp. 600-606,1984.
Printed in Great Britain
0021-8928/84 $10.00+0.00

APPLICATION OF DUALITY METHODS IN PROBLEMS OF OPTIMIZING THE SHAPE OF ELASTIC BODIES*

L.V. PETUKHOV and S.I. REPIN

A method is proposed for obtaining estimates of the magnitude of the global extremum in plate and three-dimensional body shape-optimization problems. This enables an estimate to be made of the ultimate possibilities of optimization. In certain cases, a control is constructed successfully for which the values of the objective functional will be close, and sometimes even equal, to the magnitude of the global extremum.

1. **Free vibrations of thin plates.** Let there be a domain $\Omega \in R^2$ with piecewise-smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. The frequency ω of free vibrations of a plate of thickness h is given by the following relations:

$$\omega^2 = \min_{w \in V} \Phi(h, w); \quad \Phi(h, w) = \Pi(h, w)/T(h, w) \tag{1.1}$$

$$\Pi(h, w) = \int_\Omega Dh^3 \psi(x, y)\, d\Omega; \quad T(h, w) = \int_\Omega \rho h \varphi(x, y)\, d\Omega$$

$$D = E/(12(1 - \nu^2)), \quad \varphi = w^2, \quad \psi = (\Delta w)^2 - 2(1 - \nu)\cdot$$
$$\cdot (w_{,xx} w_{,yy} - w^2_{,xy})$$
$$V = \{v \mid v \in W_2^2(\Omega); \quad v = v_{,n} = 0 \text{ on } \Gamma_1, \quad v = 0 \text{ on } \Gamma_2\}$$

Here E is Young's modulus, ν is Poisson's ratio, ρ is the density, w is the deflection, x, y are Cartesian coordinates of a point, and $v_{,n}$ denotes the derivative along the normal to the contour Γ. The optimization problem is as follows: it is required to find h^* and w^* such that

$$\Phi(h^*, w^*) = \sup_{h \in H} \inf_{w \in V} \Phi(h, w) \tag{1.2}$$

$$H = \left\{h \in L_\infty(\Omega) \,\middle|\, \int_\Omega h\, d\Omega = h_3 \operatorname{mes} \Omega, \ h_1 \leqslant h \leqslant h_2\right\}$$

$$h_2 > h_3 > h_1 > 0$$

where $\operatorname{mes}\Omega$ denotes the Lebesgue measure of the domain Ω.

It is known that in problems of this kind, the existence of generalized solutions /5,6/ is possible in addition to piecewise-smooth solutions /1-4/. Moreover, problem (1.2) is non-convex; consequently, different numerical algorithms only result in locally optimal solutions /7,8/. However, attempts can be made to find the function $h \in H$ for which the value of the objective functional is less than the supremum by a certain small quantity ε. For this it is necessary to estimate the value of the supremum, as can be done by using the dual problem.

The following problem is called the dual of the original /9/: Find h^*, w^* such that

$$\Phi(h^*, w^*) = \inf_{w \in V} \sup_{h \in H} \Phi(h, w) \tag{1.3}$$

The following inequalities are obviously valid

$$\sup_{w \in V} \inf_{h \in H} \Phi(h, w) \leqslant \inf_{h \in H} \sup_{w \in V} \Phi(h, w) \tag{1.4}$$

and can be used to construct upper bounds for the magnitude of the supremum in problem (1.2). We use the notation

$$Q_0 = \sup_{h \in H} \Phi(h, w_0), \ w_0 \in V \tag{1.5}$$

---

*Prikl.Matem.Mekhan.,Vol.48,5,830-838,1984

By virtue of (1.3) and (1.4), here Q_0 is the upper bound. To find Q_0 it is necessary to construct a function h_0 such that $\Phi(h_0, w_0) = Q_0$.

Assertion 1. The intermediate regime $h_1 < h < h_2$ is impossible in problem (1.5).

Proof. Let $h_0 = h_0(x, y)$ be the solution of (1.5). We assume that there is a subdomain $\Omega_1 \subset \Omega$, $\operatorname{mes}\Omega_1 > 0$ such that $h_1 < h_0 < h_2$ if $(x, y) \in \Omega_1$. We choose Ω_1' and Ω_1'' such that $\Omega_1 = \Omega_1' \cup \Omega_1''$, $\Omega_1' \cap \Omega_1'' = \varnothing$, $\operatorname{mes}\Omega_1' = \operatorname{mes}\Omega_1''$, and we construct the function h_δ. We set $h_\delta = h_0$ in $\Omega \setminus \Omega_1$; $h_\delta = h_0 - \delta$ in Ω_1'; $h_\delta = h_0 + \delta$ in Ω_1'' ($\delta = \text{const}$).
It can be shown that

$$\Phi(h_\delta, w_0) - \Phi(h_0, w_0) = [-\delta\Delta + 3\delta^2 T\Pi_2 + T\Pi_3\delta^3] \times [T^2 - \delta T T_1]^{-1} \tag{1.6}$$

$$\Delta = T_1\Pi - 3\Pi_1 T,\ \Pi = \Pi(h_0, w_0),\ T = T(h_0, w_0)$$

$$T_1 = \int_{\Omega_1'} \varphi\, d\Omega - \int_{\Omega_1''} \varphi\, d\Omega,\quad \Pi_1 = \int_{\Omega_1'} h_0^2\psi\, d\Omega - \int_{\Omega_1''} h_0^2\psi\, d\Omega$$

$$\Pi_2 = \int_{\Omega_1} h_0\psi\, d\Omega,\quad \Pi_3 = \int_{\Omega_1'} \psi\, d\Omega - \int_{\Omega_1''} \psi\, d\Omega$$

If $\Delta = 0$ then for small δ the right side of (1.6) is positive. If $\Delta \neq 0$, then by selecting $\delta > 0$ or $\delta < 0$ depending on the sign of Δ, we obtain that the right side of (1.6) can be made positive for sufficiently small δ. This contradicts the fact that h_0 is a solution of (1.5).

Corollary. We introduce a new control function $\mu = (h_2 - h)/(h_2 - h_1)^{-1}$ ($\mu = 1$ if $h = h_1$ and $\mu = 0$ if $h = h_2$). Then

$$Q_0 = \sup_{\mu \in M} \Phi(\mu, w_0) \tag{1.7}$$

$$\Phi(\mu, w_0) = \int_\Omega (A\mu + B)\psi(w_0)\, d\Omega \times \left[\int_\Omega (C\mu + D)\varphi(w_0)\, d\Omega\right]^{-1}$$

$$A = h_1^3 - h_2^3,\ B = h_2^3,\ C = h_1 - h_2,\ D = h_2$$

Here M is the set of functions which can only take values of 0 or 1 at each point of the domain Ω and satisfies the isoperimetric condition

$$\int_\Omega (\mu h_1 + (1 - \mu) h_2)\, d\Omega = h_3 \operatorname{mes}\Omega$$

Let N denote the closure of the set M in a $*$-weak topology of the space $L_\infty(\Omega)$

$$N = \left\{\mu \in L_\infty(\Omega) \mid 0 \leqslant \mu \leqslant 1,\ \int_\Omega (\mu h_1 + (1 - \mu) h_2)\, d\Omega = h_3 \operatorname{mes}\Omega\right\}$$

Evidently

$$Q_0 = \sup_{\mu \in N} \Phi(\mu, w_0) \tag{1.8}$$

Assertion 2. Problem (1.8) has a solution, i.e., a function $\mu_0 \in N$ exists on which the functional Φ achieves its exact upper bound.

Proof. Let $\{\mu_n\}$ be a maximizing sequence, i.e., $\Phi(\mu_n, w_0) \to Q_0$ as $n \to \infty$. It is uniformly bounded, consequently, a subsequence can be extracted that is also maximizing and convergent to a certain function $\mu_0 \in N$ in $*$-weak topology L_∞ /9/. Passing to the limit in (1.8), we have $\Phi(\mu_n, w_0) \to \Phi(\mu_0, w_0)$, $n \to \infty$, from which there follows that $\Phi(\mu_0, w_0) = Q_0$.

Assertion 3. The function μ_0 that yields a maximum of the functional $\Phi(\mu, w_0)$ takes only the two values 0 and 1, where if Ω_0, Ω_1 are, respectively, the subdomains where $\mu_0 = 0$ and $\mu_0 = 1$, then

$$R(x, y) \geqslant R(\xi, \eta),\ \forall (x, y) \in \Omega_1,\ \forall (\xi, \eta) \in \Omega_0 \tag{1.9}$$

$$R(u, v) = \psi(u, v) - Q_0\varphi(u, v)/(h_1^2 + h_1h_2 + h_2^2)$$

The proof that μ_0 does not take on intermediate values is analogous to the proof of Assertion 1.
We consider two sequences of subdomains $\{\omega_k^+\} \in \Omega_1$, $\{\omega_k^-\} \in \Omega_0$; $\operatorname{mes}\omega_k^+ = \operatorname{mes}\omega_k^-$ ($k = 1, 2, \ldots$), where

$$\omega_k^+ \in S(x, y, \rho_k),\ \omega_k^- \in S(\xi, \eta, \rho_k),\ \rho_k \to 0,\ k \to \infty$$

$$S(u, v, \rho) = \{(x, y) \in \Omega \mid (x - u)^2 + (y - v)^2 \leqslant \rho^2\}$$

We construct the sequence of controls

$$\mu_k = \begin{cases} \mu_0 \text{ in } \Omega \setminus (\omega_k^+ \cup \omega_k^-) = \Omega_k \\ 0 \text{ in } \omega_k^+ \\ 1 \text{ in } \omega_k^- \end{cases}$$

If μ_0 is a solution of problem (1.8), the inequality

$$\Phi(\mu_0, w_0) - \Phi(\mu_k, w_0) \geqslant 0$$

should be satisfied for any k, from which it follows that

$$T'A(a_k^+ - a_k^-) - \Pi'C(b_k^+ - b_k^-) \geqslant L(a_k^+b_k^+, a_k^+b_k^-, a_k^-b_k^+, a_k^-b_k^-) \tag{1.10}$$

$$a_k^+ = \int_{\omega_k^+} \psi\, d\Omega, \quad a_k^- = \int_{\omega_k^-} \psi\, d\Omega, \quad b_k^+ = \int_{\omega_k^+} \varphi\, d\Omega, \quad b_k^- = \int_{\omega_k^-} \varphi\, d\Omega$$

$$T' = \int_{\Omega_k} (C\mu_0 + D)\varphi\, d\Omega, \quad \Pi' = \int_{\Omega_k} (A\mu_0 + B)\psi\, d\Omega$$

(L is a linear function of its arguments). Dividing both sides of inequality (1.10) by $\operatorname{mes}\omega_k^+$ and passing to the limit as $k \to \infty$, we obtain (1.9).

Remark. Inequality (1.9) enables us to construct the function μ_0. We choose a certain value of Q, we substitute it into (1.9) in place of Q_0, and we construct a corresponding function μ. We calculate $Q^* = \Phi(\mu, w_0)$; Q_0 corresponds to the greatest of those values of Q for which $Q = Q^*$, and can be found by standard methods for finding the roots of transcendental equations.

Examples. We consider a rectangular plate with sides a and b, and let $h = h(x)$. The sides $y = 0, b$ of the plate are freely supported, while $x = 0, a$ are free. Problem (1.1) is reduced to the following

$$\omega^2 = \inf \Phi(h, F); \quad \Phi(h, F) = \Pi/T \tag{1.11}$$

$$\Pi = \int_0^a (m_{22}F'' - \alpha^2 m_{11}F + 2m_{12}\alpha F')\, dx, \quad T = \int_0^a hF^2 dx$$

$$m_{22} = Dh^3(F'' - \alpha^2\nu F), \quad m_{11} = Dh^3(\nu F'' - \alpha^2 F), \quad m_{12} = Dh^3(1-\nu)F'$$

$$U = \{u \in W_2^2[0, a] \mid u(0) = u(a) = 0\}, \quad \alpha = n\pi/b$$

It is necessary to find a quantity $h_* \in H_2$ such that

$$\omega^2(h_*) = \sup_{h \in H_2} \omega^2(h) \tag{1.12}$$

$$H_2 = \left\{h \in L_\infty[0, a] \mid h_1 \leqslant h \leqslant h_2; \ \int_0^a h\, dx = h_0 a\right\}$$

Using (1.9) we obtain the following estimate of the supremum ($a = b = 1$):

$$Q_0 = \pi^4(h_1^2 + h_2^2 + h_1h_2 - h_1^2h_2 - h_2h_1^2)$$

It turns out that elements exist in the set H_2 for which the value of the objective functional is close to Q_0. To show this, the formulation of problem (1.12) must be altered somewhat. As already mentioned, problem (1.12) may not even have a solution. To eliminate this singularity, it is necessary to perform a G-closure of the boundary value problem (1.11) /10/. Therefore, a new expanded (relaxation) problem of optimization can be constructed whose solution exists and can be approximated to the functional with any degree of accuracy by elements of the original set H_2. If we limit ourselves to an examination of controls taking on just two values h_1 and h_2 then the G-closure of (1.11) results in the following variational problem

$$\omega^2 = \inf \Phi(\lambda, F); \quad \Phi(\lambda, F) = \Pi\left[\int_0^a (\lambda h_1 + (1-\lambda)h_2)F^2 dx\right]^{-1} \tag{1.13}$$

$$m_{22} = b_{22}(F'' - \alpha^2\nu F), \quad m_{12} = 2b_{12}F', \quad m_{11} = \nu b_{22}F'' - \alpha^2 b_{11}F$$

$$b_{22} = Dh_1^3h_2^3[\lambda h_2^3 + (1-\lambda)h_1^3]^{-1}, \quad b_{12} = D(1-\nu)(\lambda h_1^3 + (1-\lambda)h_2^3)$$

$$b_{11} = (1+\nu)b_{12} + \nu^2 b_{22}$$

The expanded optimization problem is to seek a $\lambda_* \in N$ such that

$$\omega^2(\lambda_*) = \sup_{\lambda \in N} \omega^2(\lambda) \tag{1.14}$$

Here λ is a new control function which is related to h as follows. If $\lambda = 1$ in a certain interval $D_1 = [x_1, x_1 + \Delta]$, then $h(x) = h_1$, $x \in D_1$; if $\lambda(x) = 0$, then $h(x) = h_2$; $x \in D_1$, if $\lambda = 0.5$, say, then this corresponds to an infinitely frequent uniform alternation of the thicknesses h_1 and h_2.

To obtain the lower bound of the supremum we take a certain value $\lambda(x) \in \Lambda$ and solve problem (1.13). We set $\lambda(x) = 0.5$. We calculate the corresponding value of the frequency ω_*. We let ω_p denote the frequency for a constant thickness plate of the same weight, and let $a = b = 1, h_3 = 1$. Then for $h_1 = 0.7$ and $h_2 = 1.3$, we have $\omega_*/\omega_p = 1.12$, $q_0/\omega_p = 1.18$; for $h_1 = 0.5$, $h_2 = 1.5$ we have $\omega_*/\omega_p = 1.30$, $q_0/\omega_p = 1.38$, and for $h_1 = 0.3$, $h_2 = 1.7$ we have $\omega_*/\omega_p = 1.54$, $q_0/\omega_p = 1.65$, where $q_0 = \sqrt{Q_0}$. It is seen that values of the gains are fairly close to the upper bound of the supremum.

We examine the following sequence of controls $\{h_k\} \in H_2, k = 1, 2, \ldots$, $h = h_1$ if $x \in [(i-1)\Delta, i\Delta)$; $h = h_2$, if $x \in [i\Delta, (i+1)\Delta)$, $\Delta = 1/k$, $i = 1, 2, \ldots, k-1$. We let ω_k denote the value of the frequency corresponding to the function h_k. By virtue of the properties of the expanded problem, we have $\omega_k \to \omega_*$ as $k \to \infty$. Hence, for sufficiently large k the ratio ω_k/ω_p is close to ω_*/ω_p while the value of the objective functional for the control $h_k \in H_2$ is close to the magnitude of the global supremum.

Analogous estimates can be obtained in other problems also, when the control depends on one coordinate. For instance, we consider a long rectangular plate $(a \gg b)$ with freely supported edges and we let $h = h(x)$. As w_0 we select the function

$$w_0 = F(x)\sin\alpha y;\quad F(x) = \begin{cases} \sin\dfrac{\pi x}{2z}, & x \leqslant z \\ 1, & z < x < a - z \\ \sin\dfrac{\pi(b-x)}{2z}, & x \geqslant a - z \end{cases}$$

In this case

$$\frac{Q_0}{\omega_p^2} = \frac{h_2^3[1+{}^{1}/_{4}z^{-2}]^2 - 2h_1^3 + a[h_1^2 + h_2^2 + h_1h_2 - h_1^2h_2 - h_2h_1^2]z^{-1}}{h_1 - 2h_2 + az^{-1}}$$

$$\frac{\omega_*^2}{\omega_p^2} = \frac{b_{11}a^2b^{-2} + 2(\nu b_{22} + b_{12}) + b_{22}b^2a^{-2}}{h_3^2(ba^{-1} + ab^{-1})^2}$$

For instance, for $h_1 = 0.5$, $h_2 = 1.5$, $h_3 = 1.0$, $b/a = 20$ we obtain $Q_0/\omega_p^2 = 2.28$, $\omega_*^2/\omega_p^2 = 1.61$, and $\omega_*^2/Q_0 = 0.71$. Therefore, the value of the frequency obtained is not less than 84% of the upper bound. The problem can also be considered for an annular plate of radii r and $R\,(R > r)$ when the thickness depends only on the angular coordinate α. It turns out that for $r/R >$ 0.7 – 0.8 a plate with uniformly arranged radial ribs will also ensure a solution close to the globally optimal one.

In two-dimensional problems, when $h = h(x, y)$, inequality (1.9) permits rapid construction of h_0 and calculation of Q_0 for a given function ω_0. Thus, $Q_0/\omega_p^2 = 2.21$ is obtained for the case when $h_1 = 0.5$, $h_2 = 1.5$, $h_3 = 1$ (all sides of a square plate freely supported), while we have $Q_0/\omega_p^2 = 1.45$ for $h_1 = 0.7$, $h_2 = 1.3$, $h_3 = 1.0$.

Therefore, the upper boundary of the magnitude of the parameter being optimized can be established at once. It should be kept in mind that the estimate obtained by such a method can turn out to be quite excessive either because of the unsuccessful choice of w_0 or because of the presence of an unavoidable gap between the direct and dual problems in inequality (1.4).

2. Formulation of optimization problems for a three-dimensional elastic medium. Let the domain $\Omega \subset R^3$ be filled by an elastic medium consisting of two materials that are characterized by the elastic constant tensors $\theta a = \{\theta a_{ijkl}\}$, the densities $\theta\rho$, and the torsion yield points $\theta\sigma_0$, where $\theta = \theta_0 > 0$ $(\theta_0 \ll 1)$ or $\theta = 1$. Here and everywhere later, the Roman subscripts run through the values 1 - 3. The body force vector $f(x) = \{f_i(x)\}$ is given in the domain Ω, where $x = \{x_i\}$ are Cartesian coordinates of the points in Ω. We consider the boundary of Ω, which we shall denote by Γ, as consisting of two parts Γ_u and Γ_F $(\Gamma = \Gamma_u \cup \Gamma_F)$, where the displacement vector $u = \{u_i\} = 0$ is given on the former, and the surface loads vector $F = \{F_i\}$ on the latter. The quantities $a_{ijkl}, \rho, \sigma_0, \theta_0$ are assumed to be independent of the coordinates x_i.

We expand the initial formulation of the problem somewhat. We assume that there is a hypothetical inhomogeneous elastic medium for which $0 < \theta_0 \leqslant \theta(x) \leqslant 1$.

We now specify the quantities introduced above. Let Ω be an intrinsically regular domain /11/. We introduce the sets

$$\theta = \{\Theta \in L_\infty(\Omega) \mid 0 < \theta_0 \leqslant \theta(x) \leqslant 1\}$$
$$U = \{u \mid u_i \in H^1(\Omega),\ u_i = 0 \text{ on } \Gamma_u\}$$

where $H^1(\Omega)$ is the S.L. Sobolev space /12/. We will assume that

$$f_i \in L_2(\Omega),\ F_i \in L_2(\Gamma_F)$$

For any fixed $\theta \in \Theta$ the displacement $u \in U$ of the points of Ω is governed by the integral identity /12/

$$\int_\Omega (\theta a_{ijkl}\varepsilon_{ij}\varkappa_{kl} - v_i f_i)\, dx - \int_{\Gamma_F} F_i v_i\, d\Gamma = 0 \tag{2.1}$$
$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \quad \forall v \in U, \quad \varkappa_{kl} = (v_{k,l} + v_{l,k})/2$$

where $\varepsilon = \{\varepsilon_{ij}\}$, $\varkappa = \{\varkappa_{kl}\}$ are strain tensors $(\cdot)_{,i}$ is the derivative with respect to the coordinate x_i, summation between 1 and 3 is assumed over the repeated subscripts here and everywhere later. If the solution u is found, then the stress tensor $\sigma = \{\sigma_{ij}\} = \{\theta a_{ijkl}\varepsilon_{kl}\}$ can be determined.

The constraint $b_{ijkl}\sigma_{ij}\sigma_{kl} \leqslant \theta^2\sigma_0^2$ on the stress for almost all $x \in \Omega$ is often encountered in optimal design problems, from which a constraint on the strain evidently results

$$c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \leqslant \sigma_0^2 \tag{2.2}$$

for almost all $x \in \Omega$, where $b = \{b_{ijkl}\}, c = \{c_{ijkl}\}$ are tensors determining the invariants in the constraints on the stress and strain, $c_{ijkl} = b_{pqrs}a_{ijpq}a_{klrs}$.

We will now formulate two optimal control problems: the problem of minimizing the mass of material under constraints on the stress (problem W)

$$\inf \int_\Omega \theta\rho\, dx \tag{2.3}$$

for $\forall\theta \in \Theta, \forall u \in U$ satisfying (2.1) and (2.2), and the problem of minimizing the elastic strain energy under constraints on the material mass (problem P)

$$\inf \int_\Omega \frac{1}{2}\theta a_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\, dx \tag{2.4}$$

for $\forall u \in U$ satisfying the equality (2.1) and

$$\forall\theta \in \vartheta = \left\{\theta \in \Theta \,\middle|\, \int_\Omega \theta\rho\, dx = \rho_0 \operatorname{mes}\Omega, \quad \theta_0 < \rho_0 / \rho < 1\right\}$$

3. Dual estimate in problem W.

We construct the Lagrange functional

$$L(u, \theta, v, \mu) = \int_\Omega (\theta\rho + \theta a_{ijkl}\varepsilon_{ij}\varkappa_{kl} + \mu c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \tag{3.1}$$
$$v_i f_i - \mu\sigma_0^2)\, dx - \int_{\Gamma_F} F_i v_i\, d\Gamma, \quad \forall u, v \in U, \quad \forall\theta \in \Theta$$
$$\forall\mu(x) \in \mathrm{M} = \{\mu \in L_\infty(\Omega) \mid \mu \geqslant 0\}$$

We examine two extremal problems for the functional L.

The first

$$\inf L^\circ(u, \theta), \ \forall u \in U, \ \forall\theta \in \Theta; \ L^\circ(u, \theta) = \sup L, \ \forall v \in U, \ \forall\mu \in \mathrm{M}$$

is equivalent to problem W.

The second

$$\sup L_0(v, \mu), \ \forall v \in U, \ \forall\mu \in \mathrm{M}; \ L_0(v, \mu) = \inf L, \ \forall\varepsilon_{ij} \in L_2(\Omega), \ \forall\theta \in \Theta$$

is the dual to problem W and can be used for the lower bound of the value (2.3).

Indeed, from the evident inequalities

$$L_0(v, \mu) \leqslant \sup_{v \in U, \mu \in \mathrm{M}} L_0(v, \mu) \leqslant \sup_{v \in U, \mu \in \mathrm{M}} \inf_{u \in U, \theta \in \Theta} L \leqslant \tag{3.2}$$
$$\inf_{u \in U, \theta \in \Theta} \sup_{v \in U, \mu \in \mathrm{M}} L = \inf_{u \in U, \theta \in \Theta} L^\circ(u, \theta)$$

it follows that the value of $L_0(v, \mu)$ yields the lower bound for (2.3).

We will now construct the functional $L_0(v, \mu)$. For each fixed v and μ the integrand in the first integral in (3.1) is a convex function of ε_{ij}, hence, it is necessary to minimize the function

$$\varphi(\varepsilon) = \mu c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \theta a_{ijkl}\varepsilon_{ij}\varkappa_{kl}$$

In the general case the tensor μc is not positive-definite, but only positive, consequently, the necessary and sufficient condition for a minimum

$$2\mu c_{ijkl}\varepsilon_{kl} + \theta a_{ijkl}\varkappa_{kl} = 0, \quad \forall v \in U, \quad \forall\mu \in \mathrm{M} \tag{3.3}$$

cannot possibly be solved directly for ε_{kl}.

In fact, we examine the constraint on the shear stress intensity (the Mises yield condition) /13/ for an isotropic material as the strength condition. Then

$$\varphi(\varepsilon) = 4G^2\mu\,[(\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{22} - \varepsilon_{33})^2 + (\varepsilon_{33} - \varepsilon_{11})^2 + \tag{3.4}$$

$$6\,(\varepsilon_{12}^2+\varepsilon_{23}^2+\varepsilon_{31}^2)]/6+\theta a_{ijkl}\varkappa_{kl}\varepsilon_{ij}$$

Conditions (3.3) for (3.4) take the following expanded form

$$4G^2\mu\,(3\varepsilon_{ii}-\varepsilon_{11}-\varepsilon_{22}-\varepsilon_{33})/3+\theta a_{iikl}\varkappa_{kl}=0 \tag{3.5}$$
$$4G^2\mu\varepsilon_{ij}+\theta a_{ijkl}\varkappa_{kl}=0\ (i\neq j)$$

from which it follows that if $\varkappa_{11}+\varkappa_{22}+\varkappa_{33}\neq 0$, then $\inf_\varepsilon \varphi(\varepsilon)=-\infty$, as is achieved for $\varepsilon_{11}=\varepsilon_{22}=\varepsilon_{33}=\pm\infty$. Therefore, the function $\varphi(\varepsilon)$ only takes finite values under the condition

$$\varkappa_{11}+\varkappa_{22}+\varkappa_{33}=0 \tag{3.6}$$

Under condition (3.6) the solution (3.5) can be found in the form

$$\varepsilon_{ii}=D-\theta a_{iikl}\varkappa_{kl}/(4G^2\mu) \tag{3.7}$$
$$\varepsilon_{ij}=-\theta a_{ijkl}\varkappa_{kl}/(4G^2\mu)\ (i\neq j)$$

where D is an arbitrary constant. Substituting (3.7) into (3.4) and taking (3.6) into account, we obtain

$$\inf_\varepsilon \varphi(\varepsilon)=\frac{\theta^2}{\mu}\Big[\frac{1}{6}(\varkappa_{11}-\varkappa_{22})^2+\frac{1}{6}(\varkappa_{22}-\varkappa_{33})^2+ \tag{3.8}$$
$$\frac{1}{6}(\varkappa_{33}-\varkappa_{11})^2-\varkappa_{11}^2-\varkappa_{22}^2-\varkappa_{33}^2-\varkappa_{12}^2-\varkappa_{23}^2-\varkappa_{31}^2\Big]=-\frac{\theta^2}{\mu}\psi(x)$$
$$\psi(\varkappa)=d_{ijkl}\varkappa_{ij}\varkappa_{kl}$$

Analysis shows that the tensor $d=\{d_{ijkl}\}$ is positive-definite. At points at which $\mu=0$ the function $\varphi(\varepsilon)$ degenerates into a linear function and $\inf\varphi(\varepsilon)=-\infty,\ \forall\varepsilon_{ij}\in L_2(\Omega)$.

We substitute (3.8) into (3.1), then

$$\inf_{\varepsilon_{ij}\in L_2(\Omega)} L=\int_\Omega\Big[\theta\rho-\frac{\theta^2}{\mu}\psi(\varkappa)-v_if_i-\mu\sigma_0^2\Big]dx-\int_{\Gamma_F}F_iv_i\,d\Gamma$$

after which we find

$$L_0(v,\mu)=\int_\Omega[\omega(\varkappa,\mu)-v_if_i-\mu\sigma_0^2]\,dx-\int_{\Gamma_F}F_iv_i\,d\Gamma \tag{3.9}$$
$$\omega(\varkappa,\mu)=\begin{cases}\rho-\psi(\varkappa)/\mu, & \mu\leqslant(1+\theta_0)\psi(\varkappa)/\rho\\ \theta_0\rho-\theta_0^2\psi(\varkappa)/\mu, & \mu\geqslant(1+\theta_0)\psi(\varkappa)/\rho\end{cases}$$

For fixed v and :_ the value of $L_0(v,\mu)$ yields a lower bound of the infimum of the initial problem (see the inequalities (3.2)). To improve this estimate, (3.9) should be maximized. The functional (3.9) is independent of the value $\theta>\theta_0$ and $\theta<1$, therefore, the presence of two materials with $\theta=\theta_0$ and $\theta=1$ could be assumed in Sec. 2.

Example. We consider a cube with edge e loaded on two opposite faces $x_3=0$ and $x_3=e$ by compressive forces of constant intensity F, which satisfies the inequalities $\theta_0\leqslant F(\sqrt{3}\sigma_0)^{-1}\leqslant 1$ obtained from the equal-strength condition in the sense of the Mises condition /3/ for $\theta=\theta_0$ and $\theta=1$. The optimal control and the corresponding mass of the material are

$$\theta=F(\sqrt{3}\sigma_0)^{-1},\quad m_0=\rho e^3F(\sqrt{3}\sigma_0)^{-1}$$

We now obtain the lower bound by using (3.9). We set $v_1=-\alpha x_1/2, v_2=-\alpha x_2/2,\ v_3=\alpha x_3$, then condition (3.6) will be satisfied $\psi(\varkappa)=3\alpha^2/4$ while $L_0(v,\mu)=e^3[\omega(\varkappa,\mu)-\mu\sigma_0^2+F\alpha]$, from which we find the dual estimate for

$$m^c=\sup_{\mu>0}\sup_{\alpha\in R}L_0(v,\mu)=\rho e^3[\theta_0+F^2(\sqrt{3}\sigma_0)^{-2}](1+\theta_0)^{-1}$$

Comparing m_0 and m^c, we see that $m^c\leqslant m_0$ for all allowable F, where the dual estimate is good either for small or for large allowable F. The dual estimate can be improved if (3.9) is maximized.

4. Dual estimate in problem *P*. We construct the Lagrange functional

$$L(u,\theta,v,\mu)=\int_\Omega\Big(\frac{1}{2}a_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\theta a_{ijkl}\varepsilon_{ij}\varkappa_{kl}- \tag{4.1}$$
$$v_if_i+\mu\theta\rho-\mu\rho_0\Big)dx-\int_{\Gamma_F}F_iv_i\,d\Gamma,\ \forall u,v\in U,\ \forall\theta\in\vartheta,\ \forall\mu\in R$$

If the sets Θ and M in Sec. 3 are replaced, respectively, by the sets ϑ and R, and analogous reasoning is presented, then the dual problem

$$\sup L_0(v, \mu),\ \forall v \in U,\ \forall \mu \in R \tag{4.2}$$

and inequalities analogous to (3.2) can be obtained.

Let us construct the functional $L_0(v, \mu)$. For each fixed v and μ the integrand in the first integral in (4.1) is a strongly convex function of the component ε_{ij}, consequently, the necessary and sufficient condition for the minimum of L with respect to ε_{ij} is

$$\varepsilon_{ij} = -\varkappa_{ij} \tag{4.3}$$

Substituting (4.3) into (4.1), we obtain

$$\inf_{\varepsilon_{ij}\in L_2(\Omega)} L = \int_\Omega \left(-\frac{1}{2}\theta a_{ijkl}\varkappa_{ij}\varkappa_{kl} - v_i f_i + \mu\theta\rho - \mu\rho_0\right) dx - \int_{\Gamma_F} F_i v_i \, d\Gamma$$

from which we find

$$L_0(v, \mu) = \int_\Omega [\omega(\varkappa, \mu) - v_i f_i - \mu\rho_0]\, dx - \int_{\Gamma_F} F_i v_i \, d\Gamma \tag{4.4}$$

$$\omega(\varkappa, \mu) = \begin{cases} \mu\rho - a_{ijkl}\varkappa_{ij}\varkappa_{kl}/2, & \mu \leqslant a_{ijkl}\varkappa_{ij}\varkappa_{kl}/(2\rho) \\ \mu\theta_0\rho - \theta_0 a_{ijkl}\varkappa_{ij}\varkappa_{kl}/2, & \mu \geqslant a_{ijkl}\varkappa_{ij}\varkappa_{kl}/(2\rho) \end{cases}$$

Exactly as in Sec. 3, here L_0 is independent of intermediate values of θ.

Example. We again examine a cube with edge e loaded on two opposite faces $x_3 = 0$ and $x_3 = e$ by compressive forces of constant intensity F. We set $\theta = \rho_0/\rho$, then the corresponding elastic strain energy equals

$$\Pi_0 = \rho e^3 F^2/(2E\rho_0)$$

We now obtain the lower bound by using (4.4). We set $v_1 = 0, v_2 = 0, v_3 = \alpha x_3$, $\alpha = \mathrm{const}$, then only the strain $\varkappa_{33} = \alpha$ will be different from zero; $L_0(v, \mu) = e^3[\omega(\varkappa, \mu) - \mu\rho_0 + F\alpha]$ from which we find the dual estimate for Π_0, namely

$$\Pi^0 = \sup_{\mu\in R}\sup_{\alpha\in R} L_0(v, \mu) = \rho e^3 F^2/(2E\rho_0)$$

Here $\Pi^0 = \Pi_0$, consequently $\theta = \rho_0/\rho$ is the optimal control.

If θ can take on only the values θ_0 or 1, then in this case the optimal control is the sliding mode. A similar solution is obtained in /3/.

REFERENCES

1. TADJBAKHSH I. and KELLER J.B., Strongest columns and isoperimetric inequalities for eigenvalues, Trans. ASME, Ser. E, Vol.29, No.1, 1962.
2. BANICHUK N.V., Optimization of Elastic Body Shape, Nauka, Moscow, 1980.
3. TROITSKII V.A. and PETUKHOV L.V., Optimization of Elastic Body Shape, Nauka, Moscow, 1982.
4. SEIRANYAN A.P., Optimal beam design with constraints on the natural vibrations frequency and the buckling force, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1976.
5. FILIPPOV A.F., On certain questions of optimal control theory, Vestnik Moscow State Univ., Ser. Matem. Mekhan., Astron., Fiz., Khim., No.2, 1959.
6. GAMKRELIDZE R.V., On optimal sliding modes, Dokl. Akad, Nauk SSSR, Vol.143, No.6, 1962.
7. ARMAND J.-L.P., Applications of the Theory of the Optimal Control of Systems with Distributed Parameters to Problems of Structure Optimization /Russian translation/, Mir, Moscow, 1977.
8. OLHOFF N., Optimal Design of Structures /Russian translation/, Mir, Moscow, 1981.
9. HSIA G., Optimization. Theory and Algorithms /Russian translation/, Mir, Moscow, 1973.
10. ZHIKOV V.V., KOZLOV S.M., OLEINIK O.A., NGOAN KHA T'EN., Averaging and G-convergence of differential operators, Usp. Matem. Nauk, Vol.34, No.5, 1979.
11. FICHERA G., Existence Theorems in Elasticity Theory /Russian translation/, Mir, Moscow, 1974.
12. DUVAUT G. and LIONS J.-L., Inequalitites in Mechanics and Physics /Russian translation/, Nauka, Moscow, 1980.
13. LUR'E A.I., Elasticity Theory, Nauka, Moscow, 1970.

Translated by M.D.F.